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THE NATURE OF LOW $T/|W|$ DYNAMICAL INSTABILITIES IN DIFFERENTIALLY ROTATING STARS

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ABSTRACT

Recent numerical simulations indicate the presence of dynamical instabilities of the f-mode in differentially rotating stars even at very low values of $T/|W|$, the ratio of kinetic to potential energy. In this Letter we argue that these may be shear instabilities which occur when the degree of differential rotation exceeds a critical value *and* the f-mode develops a corotation point associated with the presence of a continuous spectrum. Our explanation, which is supported by detailed studies of a simple shell model, offers a straightforward way of understanding all of the key features of these instabilities.

Subject headings: hydrodynamics—instabilities—gravitational waves—stars: neutron—stars: rotation

1. INTRODUCTION

Interest in dynamical instabilities of rotating stars is strong because of the prospects for detecting gravitational waves from the oscillations of neutron stars. Modelling suggests that for gravitational wave emission to be at a detectable level we require that the oscillations be unstable, and hence capable of growth.

One key factor under investigation is the effect of differential rotation. Differential rotation may arise in neutron stars in several circumstances. The first is at birth; the latest studies of rotational core collapse indicate that neutron stars will be born with strong differential rotation (Dimmelmeier, Font & Müller 2002; Ott et al 2004). Subsequent accretion of supernova fallback material or material from a companion star (Fujimoto 1993), may also drive differential rotation, at least in the surface layers. Another possibility is that oscillations may drive the star into differential rotation, via non-linear effects (Rezzolla et al 2000; Levin & Ushomirsky 2001). A differentially rotating massive neutron star may also be generated following a binary neutron star merger (Shibata & Uryu 2000). Differential rotation will be relevant provided that it can be maintained in the presence of viscosity or an internal magnetic field, which will act to bring the star into uniform rotation (Shapiro 2000).

Differential rotation has two key effects that are relevant to the study of dynamical instabilities. Firstly, differentially rotating stars exhibit a continuous spectrum, with dynamical behaviour that is distinct from the discrete normal modes of oscillation found in uniformly rotating stars. Secondly, differential rotation can lead to the occurrence of dynamical shear instabilities, unstable oscillations that do not exist in uniformly rotating systems (Balbinski 1985b; Luyten 1990a).

Before proceeding we note some points on terminology. We state that a mode is *co-rotating* if its pattern speed $\sigma_p = \sigma/m$ is positive, and *counter-rotating* if its pattern speed is negative (where the modes are assumed to behave as $\exp(-i(\sigma t - m\varphi))$). If the pattern speed of the mode matches the local angular velocity at a point in the star we state that the mode has a *corotation point*. The range of frequencies in which modes possess corotation points is termed the *corotation band*. All modes with corotation points are co-rotating, but not all co-rotating modes possess corotation points.

We will also make reference to the *degree of differential rotation*. This refers to the difference between the maximum and minimum angular velocities within a differentially rotating star. Near uniform rotation the difference is small, and the degree of differential rotation is low. The difference increases as the degree of differential rotation rises.

The majority of studies of the oscillations of differentially rotating systems have focused on the $l = m = 2$ f-mode (the so-called bar mode). As in a uniformly rotating star, the f-mode is split into co-rotating and counter-rotating branches (Fig. 1). As the ratio of kinetic to potential energy, $\beta \equiv T/|W|$, increases, the counter-rotating mode pattern speed becomes less negative, passing through zero at the point $\beta = \beta_s$. It is at this point that secular instability would arise in the presence of gravitational radiation reaction. In the absence of such mechanisms, the counter-rotating branch pattern speed continues to increase until it merges with the co-rotating branch at $\beta = \beta_d$. This merger gives rise to the well-studied dynamical bar mode instability (Shibata, Baumgarte & Shapiro 2000; New & Shapiro 2001; Yoshida et al 2002; Karino & Eriguchi 2003). The value of β_d is high (> 0.2) even for high differential rotation.

Until recently, the high β bar mode instability was the main focus of attention. Recent studies by Shibata, Karino & Eriguchi (2002, 2003) have identified dynamical bar-mode instabilities in highly differentially rotating Newtonian polytropes at values of β as low as ~ 0.01. The mechanism behind these instabilities has yet to be identified.

The presence of corotation points may be critical. In differentially rotating stars the co-rotating branch of the f-mode traverses the corotation band as β is increased (Fig. 1). In this Letter we demonstrate that the reported features of the low β instabilities can be explained if the co-rotating f-mode develops a dynamical shear instability at the point where it enters the corotation band. This mechanism has been studied in detail in a simple shell

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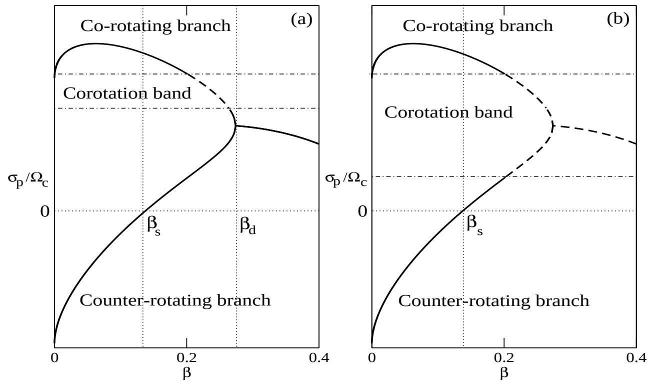

FIG. 1.— Typical dependence of pattern speed σ_p on β for the $l = m = 2$ f-modes of a differentially rotating stellar model. The quantity Ω_c is discussed in the text. Plot (a) depicts low or intermediate differential rotation, plot (b) high differential rotation. The corotation band, traversed by the co-rotating f-mode, gets wider as the degree of differential rotation rises.

model (Watts et al 2003; Watts, Andersson & Williams 2004). Dynamical instabilities associated with the presence of corotation points are also well known in accretion disc theory (Papaloizou & Pringle 1985; Goldreich, Goodman & Narayan 1986). This would however be the first occasion on which they have been observed in stellar models.

In section 2 we explain the nature of oscillations within the corotation band, and review the results of a simple model in which the corotation instability mechanism operates. In section 3 we show how the mechanism can account for the unusual features of the low β bar-mode instabilities, using the supporting evidence provided by Shibata, Karino & Eriguchi (2002, 2003).

2. THE NATURE OF OSCILLATIONS IN THE COROTATION BAND

In two previous papers (Watts et al 2003; Watts, Andersson & Williams 2004) we examined the nature of oscillations within the corotation band for a differentially rotating incompressible spherical shell, by solving the linearized Newtonian perturbation equations. We found that within the band there was a continuous spectrum that is physically distinct from the more familiar discrete normal modes. In addition, we found dynamical shear instabilities that set in when modes developed corotation points. In this section we will show that the nature of solutions within the corotation band remains the same for a more realistic stellar model.

We assume that the star is an inviscid polytrope and that the perturbations are adiabatic and non-axisymmetric with φ-dependence $\exp(im\varphi)$. We also assume that the perturbations have time dependence $\exp(-i\sigma t)$, σ being the frequency. We define the pressure P, density ρ, sound speed c, gravitational potential Φ, and angular velocity Ω. We work in the inertial frame and use cylindrical coordinates (ϖ, φ, z). Commas are used to denote partial derivatives. Combining the Euler equations, continuity equation and a polytropic equation of state for the perturbations, the Eulerian perturbation $\delta\phi \equiv \delta P/\rho$ obeys the following equation (Balbinski 1985a)

$$
\begin{aligned}
L\phi_{,zz} - \bar{\sigma}^2\phi_{,\varpi\varpi} + L\frac{\rho_{,z}}{\rho}\phi_{,z} + \bar{\sigma}^2\phi_{,\varpi}\left[\frac{L_{,\varpi}}{L} - \frac{\rho_{,\varpi}}{\rho} - \frac{1}{\varpi}\right] \\
+\phi\left[\bar{\sigma}^2\left(\frac{L}{c^2} + \frac{m^2}{\varpi^2}\right) - \frac{2m\Omega\bar{\sigma}}{\varpi}\left(\frac{\rho_{,\varpi}}{\rho} + \frac{\Omega_{,\varpi}}{\Omega} - \frac{L_{,\varpi}}{L}\right)\right] \\
-\frac{\bar{\sigma}^2 L}{c^2}\delta\Phi = 0
\end{aligned}
\tag{1}
$$

where the corotation parameter $\bar{\sigma} = m\Omega - \sigma$, the Lindblad parameter $L = 2\Omega\tilde{\Omega} - \bar{\sigma}^2$, and the vorticity $\tilde{\Omega} = 2\Omega + \varpi d\Omega/d\varpi$. Note that for a barotropic star $d\Omega/dz = d\Omega/d\varphi = 0$, so that the fluid rotates on cylinders $\Omega = \Omega(\varpi)$.

Solution of equation (1) for real σ is problematic at the corotation radius $\varpi = \varpi_c$ where $\bar{\sigma} = 0$, and at the Lindblad radii, $\varpi = \varpi_L$ where $L = 0$. In the neighbourhood of these points the character of the solution can be investigated using Frobenius analysis. The solution is regular at the Lindblad radii. At the corotation radius, although ϕ is continuous its first radial derivative is not. In general, there is both a finite step and a logarithmic discontinuity in the first derivative at ϖ_c (see Watts et al (2003) for more detail). The additional degree of freedom introduced by the finite step allows us to construct a continuous spectrum of "eigenfunctions" for all frequencies within the corotation band.

The continuous spectrum eigenfunctions found in this more realistic stellar model have the same mathematical character as those found in the shell model. As in this simple model, if one solves the initial value problem one finds that the collective perturbation associated with the continuous spectrum is non-singular and hence physical (Watts, Andersson & Williams (2004), see also Case (1960); Balbinski (1984)).

The dynamical shear instabilities found in the simple shell model occurred where modes crossed into the corotation band, at the point where modes merged with the continuous spectrum. Given that the nature of the continuous spectrum remains the same in the more realistic model it is legitimate to ask whether the same instability mechanism might also operate. In section 3 we will argue that the low β instabilities exhibit all of the key features of this mechanism. In order to do this we must first review some pertinent results from the shell model (Watts et al 2003; Watts, Andersson & Williams 2004).

As the degree of differential rotation increases, many of the normal mode frequencies approach the edge of the expanding corotation band. Some approach the boundary tangentially and do not cross into corotation. Those that cross into corotation at low degrees of differential rotation remain stable. Their eigenfunctions generally acquire a degree of singularity, but can be distinguished from the rest of the continuous spectrum by the fact that they do not have a finite step in the first derivative. This led us to term these solutions "zero-step solutions". We then carried out numerical time evolutions and an analytical evaluation of the initial value problem. When we produced power spectra of the time evolution data, the zero-step frequencies stood out just as clearly as the discrete normal modes outside the corotation band. The solutions that one finds when one solves the initial value

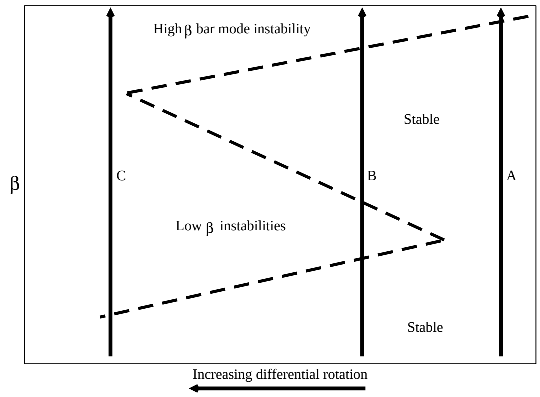

FIG. 2.— A schematic representation of Fig. 3 of Shibata, Karino & Eriguchi (2003) showing the regions of parameter space occupied by the high β bar mode and low β instabilities. The lines marked A, B and C are discussed in the text.

problem are in fact non-singular, proving that these solutions have physical relevance.

At high differential rotation zero-step solutions are still generated where modes cross into the corotation band. In addition, dynamically unstable (complex frequency) modes can be generated at the crossing point for certain rotation profiles. The eigenfunctions of these dynamically unstable modes are perfectly regular (recall that our earlier analysis applied only to real frequencies). Their pattern speeds are however in the corotation band. We refer to this instability mechanism as the corotation instability mechanism.

3. DYNAMICAL INSTABILITIES

In reviewing the low β instabilities we will focus on the j-constant rotation law used by Shibata et al. The j-constant rotation law is

$$\Omega = \frac{\Omega_c A^2}{A^2 + \varpi^2} \tag{2}$$

The parameter A^2 sets the degree of differential rotation, uniform rotation being reached in the limit as $A \to \infty$. As $A \to 0$ the degree of differential rotation increases. For a given equilibrium model, Ω_c is a function of the parameter β.

Following the convention used by Shibata et al and working in dimensionless units where $0 < \varpi < 1$, the corotation band occupies the range of frequencies

$$\frac{mA^2}{A^2+1} < \frac{\sigma}{\Omega_c} < m \tag{3}$$

Note that the corotation band widens as degree of differential rotation rises. Fig. 3 of Shibata, Karino & Eriguchi (2003),which we reproduce in schematic form in Fig. 2, illustrates the regions of parameter space occupied by the high β and low β instabilities. The following features require explanation.

For low degrees of differential rotation, the low β instabilities are not observed. Dynamical instability does not set in until high β, where the two branches of the f-mode merge (Track A, Fig. 2). For intermediate degrees of differential rotation, the co-rotating f-mode becomes unstable at low β. It stabilises again at intermediate β before merging with the other branch of the f-mode and becoming dynamically unstable again at high β (Track B, Fig. 2). For the highest degrees of differential rotation there is no stable region after the onset of the low β instability (Track C, Fig. 2). With this in mind, we propose the following explanation of these three feature.

The fact that no instabilities are observed for stars with low degrees of differential rotation (Track A, Fig. 2) suggests that, as for the shell model, this type of shear instability can only develop if the f-mode crosses into the corotation band when the degree of differential rotation exceeds a certain threshold. For the shell model we were able to determine this threshold analytically. Developing a similar criterion for the more realistic model will however be more difficult. One may, in fact, have to resort to numerical calculations. Below the threshold the f-mode passes into the corotation band but remains stable. Whether the eigenfunction is then purely regular, or possesses the singular character of the zero-step solutions that we found on the shell, is a matter for further study.

At intermediate degrees of differential rotation (Track B, Fig. 2) the co-rotating f-mode enters the corotation band and goes dynamically unstable. The lower bound of the low β instability region marks the point at which the co-rotating f-mode crosses the upper edge of the corotation band. The upper edge of the low β unstable region marks the point at which the co-rotating f-mode exits the corotation band. The system is then stable until the co-rotating branch merges with the counter-rotating branch at high β (see Fig. 1(a), which illustrates the passage of the co-rotating f-mode through the corotation band and its exit prior to mode merger).

As the degree of differential rotation rises (Track C, Fig. 2), the corotation band gets wider. For the scaling used in Fig. 1, for the j-constant law, the upper edge of the corotation band remains fixed whilst the lower edge approaches the zero-axis. The counter-rotating branch of the f-mode will now enter the corotation band before the co-rotating branch can emerge (see Fig. 1(b)). There is thus no region of stability. This raises two interesting questions. Does the counter-rotating f-mode also develop dynamical instability when it enters the corotation band? And what becomes of the traditional bar mode instability at high β? If the co-rotating f-mode never stabilises then the traditional mode merger route is excluded, but we cannot rule out the possibility of additional instabilities arising via zero-step mergers (Balbinski 1985b).

The corotation mechanism provides a simple qualitative explanation of the location the unstable regions of parameter space. It also leads to quantitative and qualitative predictions that we can test against the results of the numerical simulations reported by Shibata, Karino & Eriguchi (2002, 2003).

Firstly, all of the observed low β instabilities should have real parts of frequency that lie within the corotation band. Further communication with the authors of Shibata, Karino & Eriguchi (2002, 2003) has confirmed that all of the low β unstable modes found for the j-constant law lie within the corotation band (S.Karino, private communication).

To reproduce the shape of the instability region the co-rotating f-mode must enter the corotation band at lower values β and exit the band at higher values of β as the degree of differential rotation rises. Using numerical modelling we have confirmed that this is the case.

The third prediction is that the boundaries of the low β region of parameter space represent the points at which the f-mode enters or leaves the corotation band. For models on the lower edge of the region, we should find that $\sigma_r/\Omega_c = mA^2/(A^2+1)$, while for models on the upper edge of the low β region we should find $\sigma_r/\Omega_c = m$, where σ_r is the real part of frequency. This prediction, which would provide the most clearcut confirmation of the corotation mechanism, is however the hardest to test.

The problem lies in the growth times. Although the corotation instabilities are genuine dynamical instabilities (in that they arise in systems for which no dissipative mechanism is present), they only exhibit dynamical (short) growth times deep inside the corotation band. The shell model was sufficiently simple that we were able to track unstable modes with $|\sigma_i/\sigma_r| \sim 10^{-9}$ (σ_i being the imaginary part of frequency). It was the ability to detect these modes with very long growth times that allowed us to pinpoint the onset of instability as being near coincident with the edges of the band. Tracking slow growing instabilities using the types of numerical codes used to investigate realistic stellar models is more difficult. The code used to generate the results of Shibata et al, for example, was sensitive to modes with $|\sigma_i/\sigma_r| \sim 10^{-4}$ and larger (S.Karino, private communication). Such a limitation on the shell study would have made it impossible for us to detect the unstable modes until they were well inside the band. Conclusive tests of the third prediction will therefore require improved sensitivity to modes with long growth times. Nonetheless those low β instabilities for which growth times have been published do exhibit the expected behavior in that the growth times are shortest in the center of the band, getting longer towards the edges (see for example Fig. 4 of Shibata, Karino & Eriguchi (2002)).

Although there are other instability mechanisms associated with the corotation band (Balbinski 1985b; Luyten 1990a; Papaloizou & Pringle 1985), all have limitations as an explanation for the low β bar mode instabilities. In these scenarios the co-rotating f-mode would enter the corotation band as either a zero-step or decaying mode (although no decay is observed), merging inside the band with another zero-step or decaying mode to go unstable. What it could be merging with is however unclear: the counter-rotating f-mode is outside the band at low β, and the r-modes never enter (Karino, Yoshida & Eriguchi 2001). In addition it is not clear whether the unstable mode could retain a bar-like character if it arose through merger with another mode type; the corotation mechanism, by contrast, permits bar-like instability as the f-mode would not merge with any other mode.

The corotation mechanism may also be responsible for the low β $m = 1$ spiral instabilities observed by several authors, since the unstable modes of Pickett, Durisen & Davis (1996) and Centrella et al (2001) are all within the corotation band. This is a topic for future work.

4. CONCLUSIONS

The presence of shear instabilities in differentially rotating stars is conceptually very interesting. It is also possible that these instabilities will turn out to be of considerable astrophysical significance. These instabilities may limit the range of astrophysical rotation laws. In addition the dynamically unstable modes may lead to interesting gravitational wave signals. It is worth noting that, while it is difficult to envisage scenarios that lead to the large value of β required for the more familiar bar-mode instability, one can easily imagine situations where the low β shear instabilities will operate.

We have suggested a scenario, motivated by our analysis of the differentially rotating shell, that explains the low β f-mode instabilities observed by Shibata and collaborators. In this scenario dynamical shear instabilities arise when the co-rotating f-mode enters the corotation band and the degree of differential rotation exceeds a certain threshold value. The corotation mechanism provides a straightforward explanation for all of the unusual features of the instabilities observed in numerical simulations, and is supported by all available data. Further tests of the mechanism will require high precision studies on the onset of the instability, a task that will require codes sensitive to dynamically unstable modes with very long growth times.

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